A Transfer Function Method of Modeling Systems with Frequency-Dependent Coefficients

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The fluid forces that act on a floating body conventionally are represented in the frequency domain by frequency-dependent coefficients, the added mass, and damping. This paper describes a method of characterizing these forces in the time domain by a rational transfer function, an equivalent dynamic system that can be appended to the "dry" body dynamics. The technique is illustrated by an application to the roll motion of a flat-bottomed barge; a low-order transfer function represents the infinite-order fluid dynamic system with acceptable accuracy. The influence of the frequency-dependent coefficients on the overall system dynamics is discussed, and it is shown that multiple roll resonance peaks are possible.

Introduction

PREDICTION of the gravity wave-induced motion response of a large body in or near the sea surface is a problem of continuing interest in offshore engineering. The formulation and solution of the fluid-structure interaction is based on linear potential flow theory that assumes infinitesimal harmonic waves and motions. A variety of numerical methods[1] can be used to calculate the complex amplitudes of the fluid forces due to incident waves and body motions. Conventionally, the motion-dependent fluid forces are split into real and imaginary parts in phase with acceleration and velocity; they are referred to as added mass and damping because of the similarity between the resulting frequency domain equation of motion and the second-order differential equation which describes the motion of a mass, spring, and damper system. These hydrodynamic coefficients can be calculated for a range of frequencies, and superposition may be used to determine the body response to a summation of waves of different frequencies. Since the equation of motion contains frequency-dependent coefficients, it is valid only if the motion is harmonic; transient motions or nonlinearities in the body dynamics such as quadratic damping cannot be included in the formulation directly. Nevertheless, the fluid-structure interaction can be evaluated only in the frequency domain and it is from this representation that time-domain models of the fluid are derived.

The classical model[2] uses a convolution of the body velocity history with a force impulse function to represent the motion-dependent fluid forces. Alternatively, some equivalent lumped dynamic system whose frequency response approximates that of the fluid may be substituted for the convolution.[3] The resulting model is more convenient for some purposes; in particular, simulations are much simpler and quicker—an important factor if parametric studies of a nonlinear system subject to random waves are to be carried out. In the control theory field, similar techniques of varying degrees of sophistication have been used for some years to derive simple representations of the important features of complicated dynamical systems[4]; applications of such system identification methods in marine dynamics are still relatively infrequent.[3,5,6]

Sophisticated time-domain techniques were applied by Robins[7] and Jefferys[3] to identify discrete time models of the hydrodynamics of a wave power device moving in three interacting degrees of freedom; experimental data yielded models in which frequency responses agreed well with diffraction theory predictions. Simpler transfer function fitting techniques were applied by Booth[8] in the modeling of submarine dynamics, and a similar method has been used to characterize the fluid dynamics of an oscillating water column wave power device.[9]

In this paper, the frequency-domain transfer function fitting technique is described and applied to the roll dynamics of a flat-bottomed cargo barge for which theoretical and experimental hydrodynamic coefficients are known. If the barge dynamics are linear, the complete system model can be used in simulations to evaluate transient forces and displacements; where the body dynamics contain nonlinearities caused by quadratic damping, nonlinear moorings or fenders, the linear model of the fluid dynamics can be incorporated in an efficient nonlinear simulation.

Theory

The Classical Single-Degree-of-Freedom Model

For simplicity, the roll dynamics of the barge will be assumed to be decoupled from the other degrees of freedom. This is an excellent assumption except for the sway mode which can couple quite strongly to roll; techniques for dealing with multi-degree-of-freedom problems will be discussed later.

The roll displacements Φ are described by a second-order differential equation of the form

$$I\ddot{\Phi}(t)+C\dot{\Phi}(t)+K\Phi(t)=M_r(t)+M_d(t) \tag{1}$$

where I is the moment of inertia of the body in roll and K is the roll stiffness caused by hydrostatic forces. The damping term C is an equivalent value used to make some allowance for quadratic damping caused by vortex shedding from sharp edges such as bilge keels. It can usually be neglected if the submerged surface is smooth and free from turbulence-inducing marine growths. Exciting moments caused by the incident waves and any external forces such as crane loads are represented by $M_d(t)$. Moments generated by the dynamic reaction of the fluid to the rolling motion of the barge are denoted by $M_r(t)$. The simplest representation of these radiation forces characterizes them using terms proportional to the instantaneous roll velocity and acceleration. Equation (1) can be written as

$$(I+A)\ddot{\Phi}(t)+(B+C)\dot{\Phi}(t)+K\Phi(t)=M_d(t) \tag{2}$$

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where A is an added moment of inertia and B is a wave making damping.

If all of the parameters have known, constant values and $M_d(t)$ is specified, Eq. (2) can be solved easily in the frequency or time domains to yield complex amplitudes or time histories of roll displacement Φ. In practice, A and B are often strong functions of the frequency of the barge motion and, therefore, cannot be assumed constant in a time-domain expression.

Frequency-Dependent Coefficients

Equation (2) remains valid in the frequency domain if A and B are functions of frequency; making their dependence explicit, the equation of motion can be rewritten to relate the complex amplitudes of the moment $M_d(t)$ and response $\Phi(t)$.

$$[-\omega^2(I+A(\omega))+j\omega(C+B(\omega))+K]\Phi(j\omega)=M_d(j\omega) \quad (3)$$

Typical values of $A(\omega)$, and $B(\omega)$, normalized in the usual way, are shown in Fig. 1. They were computed for a flat-bottomed barge by a boundary integral method[10]; the same program also yields the exciting moment $M_d(j\omega)$ generated by unit amplitude waves. The relative importance of the frequency-dependent and constant parameters is a function of the body shape and inertia distribution. Equation (3) can be solved for a range of frequencies to give the response spectrum generated by any incident wave spectrum. As noted above, this technique cannot be used for transient or nonlinear problems—a time-domain model of the coefficients $A(\omega)$ and $B(\omega)$ must be developed.

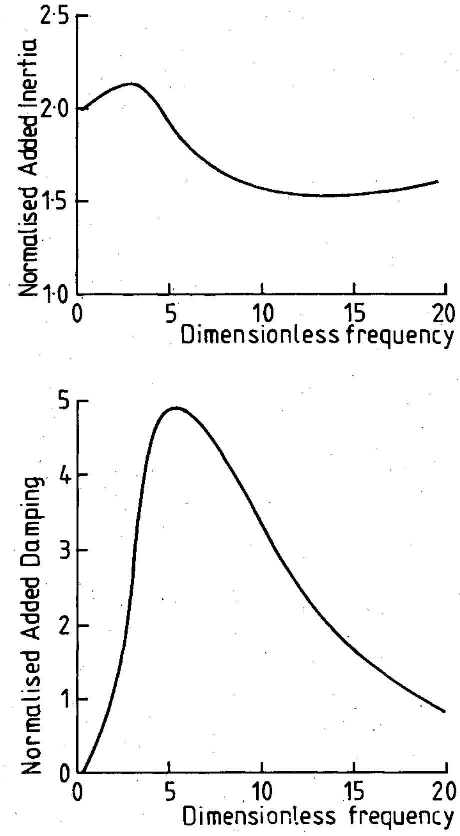

Fig. 1 Typical nondimensionalized added mass and damping.

Time-Domain Models of the Frequency-Dependent Coefficients

The coefficients $A(\omega)$ and $B(\omega)$ are the in-phase and out-of-phase parts of the complex amplitude of the radiation moment $M_r(j\omega)$ generated by the motion of the barge $\Phi(j\omega)$.

$$M_r(j\omega)=[\omega^2 A(\omega)-j\omega B(\omega)]\Phi(j\omega) \quad (4)$$

Inverse Fourier transformation of Eq. (4) and application of the convolution theorem[2] yields a time-domain expression for the radiation moment $M_r(t)$.

$$M_r(t)=-\int_0^\infty L(T)\dot{\Phi}(t-T)\mathrm{d}T-A(\infty)\ddot{\Phi}(t) \quad (5)$$

The kernel $L(t)$ is the cosine transform of the added damping divided by the frequency.[2] At high frequencies, the value of $A(\omega)$ tends to a constant value, $A(\infty)$; this term must be removed from the fluid force transfer function to ensure convergence of the inverse Fourier transform. It can be added to the barge inertia and thereby removed from the radiation problem. The value $A(\infty)$ must be computed using infinite frequency Green's functions since numerical methods become inaccurate at high but finite frequencies. Some check on the accuracy can be obtained from an interesting result[2] which shows that the mean value of the added mass is $A(\infty)$.

$$\int_0^\infty [A(\omega)-A(\infty)]\mathrm{d}\omega=0 \quad \text{or} \quad \int_0^{\omega_m} A(\omega)\mathrm{d}\omega=\omega_m A(\infty) \quad (6)$$

The convolution integral expression is substituted in Eq. (1) to yield a general equation of motion that is valid for all motions and forces, harmonic or not.

$$(I+A(\infty))\ddot{\Phi}+K\Phi+\int_0^\infty L(T)\dot{\Phi}(t-T)\mathrm{d}T=M_d(t) \quad (7)$$

Equation (7) can be solved for any exciting force time history by numerical integration. The convolution integral generates an instantaneous value for the radiation force from the past history of the roll velocity; in practice, the kernel $L(t)$ decays toward zero as t increases so the integral need only be carried out over a finite range. However, the integral must be re-evaluated at frequent intervals if numerical accuracy is to be maintained,[9] and this is extremely time consuming. In addition, the convolution model is cumbersome to use with efficient variable step length integration routines.

Lumped Model of the Radiation Force

If a lumped system can be found in which frequency and time responses are close to those of the fluid, Eq. (7) may be expressed as a high-order differential equation that is much more convenient[9] in simulation and control work. The transfer function of the approximate system $H(s)$ is the ratio of two polynomials in s, the Laplace variable. It can be shown[9] that the denominator order must be two greater than that of the numerator; a third-order model is shown here but the generalization to higher-order systems, unlikely to be necessary in this type of application, is obvious.

$$H(s)=\frac{a_1 s+a_0}{s^3+b_2 s^2+b_1 s+b_0} \quad (8)$$

Both the numerator and denominator polynomials should have their poles in the left half-plane since the fluid system is stable and the Kramers-Krönig relations[11] (known in a slightly different formulation as the Bode gain phase relations[12]) indicate that it is also minimum phase. Figure 2 shows a block diagram of the fluid and body dynamics; the fluid system forms a dynamic feedback path $H(s)$ whose input is the body motion and responds by applying radiation forces to the body dynamics, characterized by the transfer function $G(s)$.

The frequency response of the lumped model should be near that of the fluid dynamics:

$$H(j\omega)=\frac{j\omega a_1+a_0}{-j\omega^3-\omega^2 b_2+j\omega b_1+b_0}=A(\omega)+\frac{B(\omega)}{j\omega} \quad (9)$$

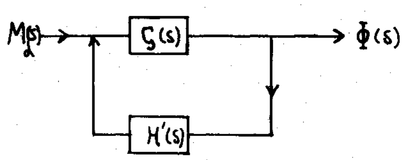

Fig. 2 Block diagram of barge and fluid dynamics.

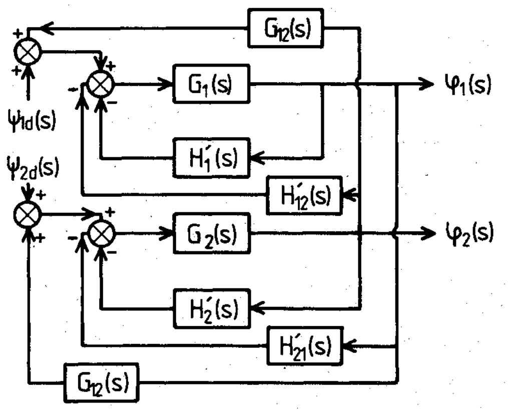

Fig. 3 Block diagram of a system with two interacting degrees of freedom.

Similarly, the time response of the system to impulse inputs should be close to that of the fluid, $L(t)$. Simple time-domain fitting techniques are not particularly successful, therefore, Eq. (9) is used to find "best" values of parameters a_i and b_i.

The transfer function $H(s)$ can be substituted for the convolution integral to give an overall transfer function $T(s)$ between exciting moment inputs and roll response outputs.

$$T(s)=\frac{\Phi(s)}{M_d(s)}=\frac{1}{(I+A(\infty)+H(s))s^2+K} \tag{10}$$

This parametric model is convenient for control work.

Alternatively, the complete system equations may be written as an array of first-order differential equations that are particularly useful in simulations.

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \dot{\phi} \\ M_r \\ x_1 \\ x_2 \end{bmatrix} \begin{matrix} =\Phi \\ -(K\phi+M_d(t)+M_r)/(I+A(\infty)) \\ =-b_2M_r+x_1 \\ -b_1M_r+a_1\ddot{\Phi}+x_2 \\ =-b_0M_r+a_1\ddot{\Phi} \end{matrix} \tag{11}$$

Subsidiary variables x_1 and x_2 have been introduced to realize a time-domain model of the transfer function.

Extension to Multi-Degree-of-Freedom Problems

In principle, the transfer function method can handle problems with any number of degrees of freedom. A separate transfer function could be fitted to each element of the (symmetric) matrix of transfer functions between body acceleration and radiation force (Fig. 3).

$$\begin{bmatrix} M_r\ (j\omega) \\ F_r\ (j\omega) \end{bmatrix}=\begin{bmatrix} A_{11}+B_{11}/j\omega & A_{12}+B_{12}/j\omega \\ A_{21}+B_{21}/j\omega & A_{22}+B_{22}/j\omega \end{bmatrix}\begin{bmatrix} \ddot{\Phi} \\ \ddot{X} \end{bmatrix}$$

$$=\begin{bmatrix} H_{11}(j\omega) & H_{21}(\omega) \\ H_{21}(j\omega) & H_{22}(j\omega) \end{bmatrix}\begin{bmatrix} \ddot{\Phi} \\ \ddot{X} \end{bmatrix} \tag{12}$$

Each transfer function $H_{ij}(j\omega)$ would have to be represented by a number of states equal to the order of its denominator in the differential equation model of the system. Hence, the number of states would be proportional to the square of the number of degrees of freedom and the simulation would be inefficient for more than two or three degrees of freedom. However, when more sophisticated methods of system identification are employed[3,7] it is found that the number of states needed is, at most, only linearly proportional to the number of degrees of freedom. This is because the interaction between modes seems to be caused by one mode "seeing" the states of the other. Each term of the transfer function matrix should, therefore, have the same denominator and only the numerators should differ. It becomes rather difficult to decide on the best approach to scaling the problem so that sensible relative weights are attached to errors between on and off diagonal terms. All in all, less ad hoc methods are to be preferred to multi-degree-of-freedom problems.

Fitting the Model to the Fluid Dynamics

The parameters a_i and b_i must be adjusted to minimize some measure of the distance between the model and fluid frequency responses. First, the order of the model must be selected in the light of the complexity of the fluid frequency response; isolated floating bodies have simple added mass and damping curves which can be represented by second- or third-order transfer functions. Only spatially separated interacting bodies possess complicated fluid dynamics; more sophisticated technique then must be applied.[7]

Approximate values of the parameters can be obtained from a Bode plot of the fluid frequency response; these values then can be refined by numerical optimization.[9] The simplest approach is to minimize the sum of the absolute distances in the complex plane between the known points on the fluid frequency response curve and the corresponding points on the transfer function frquncy response, $H(j\omega_k)$

$$\min_{a_i,\ b_c} \text{wrt } V=\sum_k \left| A(\omega_k)+\frac{B(\omega_k)}{j\omega}-H(j\omega_k)\right| \tag{13}$$

This technique works well if the model order has been chosen correctly and few parameters are to be optimized. Typically, only four or five parameters can characterize the fluid dynamics with acceptable accuracy; the numerical optimization becomes prohibitively expensive if many parameters are used. Reasonable guesses at their initial values are necessary because the optimization may converge on a spurious "local" optimum if it is started far from the true best parameter set. This technique treates errors at all points in the frequency range as equally important; a more sophisticated algorithm might minimize a weighted sum of the errors to ensure a good fit in the region where the barge response is expected to be strongest. The simple method gives a good fit over the whole frequency range in this application, therefore, a more sophisticated approach appears to be unnecessary.

Application to the Barge Roll Dynamics

The transfer function fitting technique has been applied to develop a model of the roll dynamics of a flat-bottomed barge. A comprehensive series of model tests has been carried out to validate the theoretical model of the system dynamics.[10] Theoretical predictions of dynamic response to regular and random waves accorded closely with the measured motions. Figure 1 shows the added inertia and damping coefficients, $A(\omega)$ and $B(\omega)$, calculated by a boundary integral technique. The value of $A(\infty)$ is not clear from the plot, but an initial guess, based on Eq. (6), of 14 kg-m^2 agreed closely with the figure of 13.63 kg-m^2 generated when $A(\infty)$ was fitted as an additional parameter in the optimization. A

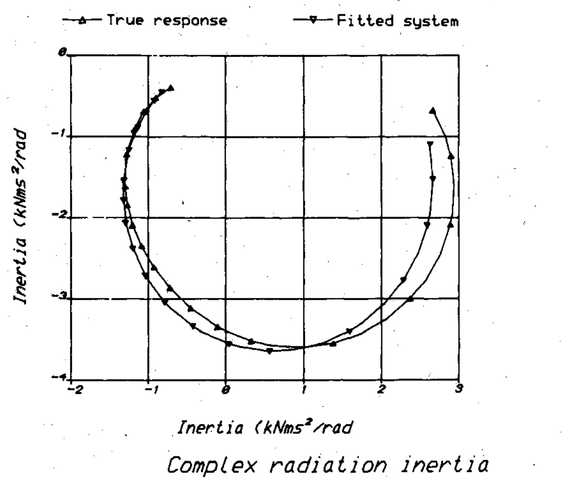

Fig. 4a Complex radiation torque generated by unit acceleration amplitude harmonic roll: exact and fitted systems.

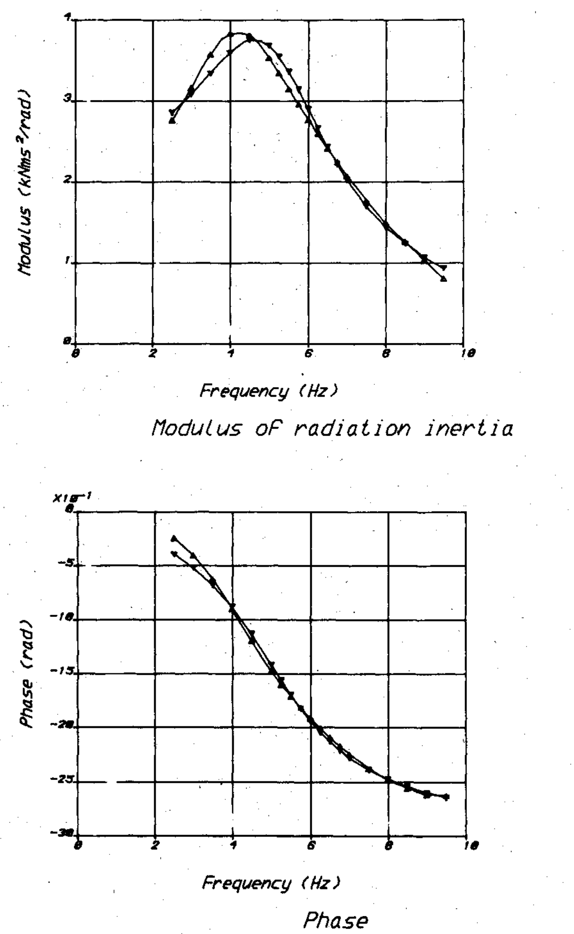

Fig. 4b Complex radiation torque generated by unit acceleration amplitude harmonic roll: amplitude and phase.

facility for calculating added inertias at infinite frequency is to be added to the numerical hydrodynamics program.

A transfer function of the form

$$H(s) = \frac{a_0}{s^2 + b_1 s + b_0} + a_1 \tag{14}$$

was assumed to allow adjustment of the initial guess of 14 kg-m² for the initial value of $A(\infty)$. Parameter values of a_0, a_1, b_0, b_1 = 64.65, −0.37, 27.79, 3.574, respectively, were found by the fitting program. Figure 4a shows the frequency response of the frequency-dependent part of $H(s)$ (full line) compared with the theoretical curve of the fluid frequency response to which it was fitted. In Fig. 4b, the same information is displayed in terms of amplitude and phase, while Figs. 5a and 5b show the corresponding added mass and damping curves; it is clear that a simple four-parameter model has characterized the fluid dynamics accurately.

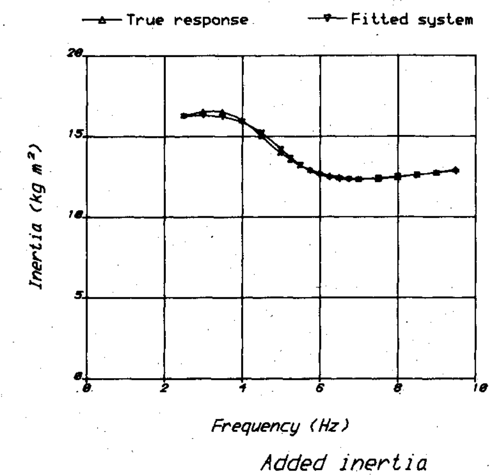

Fig. 5a Added roll inertia of exact and fitted systems.

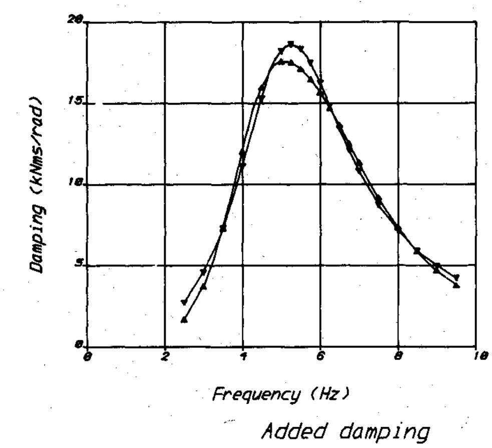

Fig. 5b Added roll damping of exact and fitted systems.

Harmonic Response of the Barge

The harmonic response of the barge to unit amplitude forcing torques can be calculated from Eq. (10) with either the transfer function or the frequency-dependent coefficients representing the fluid dynamic effects. Values of the "dry" parameters of the model barge were: moment of inertia $I = 10.9$ kg-m² and hydrostatic restoring spring $K = 913$ Nm/rad. Although the overall barge transfer function is fourth-order, it behaves very similarly to a second-order system since the frequency-independent inertia $(I + A(\infty))$ has a value of 27.14 kg-m² which is large compared to the maximum value of the frequency varying part, 4.2 kg-m². Added damping effects are only important near resonance since the system is very lightly damped, displaying a maximum dynamic magnification of 9.1 at resonance which corresponds to a damping ratio of about 0.055. The value of the frequency varying part of the added mass is only −0.27 kg-m² at resonance. Given the accuracy of the transfer function fit to the fluid dynamics, it is not surprising that the frequency responses of the barge with frequency dependent coefficients are almost indistinguishable from the system containing the transfer function model of the fluid.

Figure 6 shows a comparison between the roll transfer function measured in random wave experiments (dashed line) and the predictions of the transfer function model derived

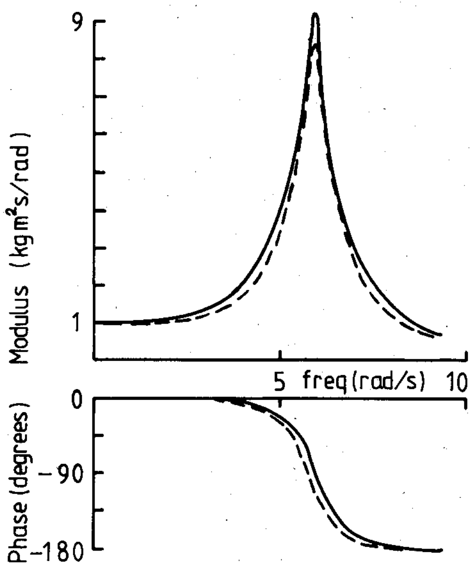

Fig. 6 Comparison between experimentally and theoretically derived added mass and damping.

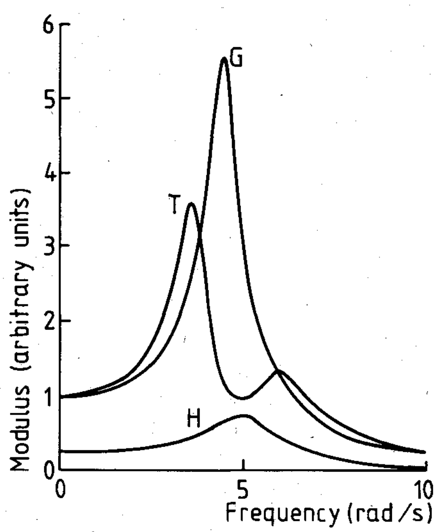

Fig. 7 Harmonic response curves displaying double peaks.

from numerial hydrodynamics. There is excellent agreement which indicates that the theory is reliable and that the lumped model is an accurate parameterization of the results of linear hydrodynamic theory.

Effect of Parameter Variations

In the example discussed previously, the effects of the frequency-dependent coefficients were masked by the total inertia of the barge. If the dry roll inertia is decreased to 1.0 kg-m^2, the frequency response develops a double peak, as shown in Fig. 7. The trough in the total system response $T(j\omega)$ corresponds to the peak in the amplitude of the radiation force. At either end of the frequency range, the effects of the radiation force die away because the motion is spring dominated at low frequencies and the radiation force transfer function tends to zero at high frequencies. The curve labeled G is the forward path response, the solid body and hydrostatic dynamics, modified by a small damping to ensure a finite resonant response.

There are no experimental reports of double-peaked roll transfer functions, probably because the dry moment of inertia required is extremely low; however, other marine systems may well possess a suitable ratio of physical-to-fluid inertia for the phenomenon to become apparent.

Conclusions

A transfer function formulation of the equations of motion of a flating body has been presented; frequency-dependent coefficients of simple form can be characterized by a low-order dynamic system. The differential equation model can be used in simulations to determine the response of the system to transient forces or its response to random waves when subject to nonlinear mooring or damping forces. In the test example, the effect of the frequency varying coefficients was, to a large extent, masked by the large dry roll inertia of the barge; it has been shown that multiple peaks in the frequency response are possible if the fluid dynamic effects are comparable with the dry dynamics of the system.

Transfer function fluid models offer an accurate characterization of the linear fluid dynamics which may be used in simulations of floating bodies with severe dynamic nonlinearities.

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